

” A Solvable Hamiltonian System ”
Integrability and Action-Angle Variables

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Abstract

We prove that the dynamical system characterized by the Hamiltonian $H = \lambda N \sum_j^N p_j + \mu \sum_{j,k}^N (p_j p_k)^{\frac{1}{2}} \{ \cos[\nu(q_j - q_k)] \}$ proposed and studied by Calogero [1,2] is equivalent to a system of *non-interacting* harmonic oscillators. We find the explicit form of the conserved currents which are in involution. We also find the action-angle variables and solve the initial value problem in simple form.

1 Introduction

Motivated by the discovery [3] that the dynamical system characterized by the Hamiltonian

$$H = \sum_{j,k}^N (p_j p_k) \exp\{-\eta(q_j - q_k)\} \quad (1)$$

is completely integrable, Calogero [1] proposed a new Hamiltonian system with the standard Poisson bracket and the Hamiltonian

$$H = \lambda N P + \mu \sum_{j,k=0}^{N-1} (p_j p_k)^{\frac{1}{2}} \{\cos[\nu(q_j - q_k)]\} \quad (2)$$

$$P = \sum_{j=0}^{N-1} p_j \quad (3)$$

The equations of motions of the dynamical variables are

$$\frac{d}{dt} q_j = \lambda N + \mu p_j^{-\frac{1}{2}} \sum_{k=0}^{N-1} p_k^{\frac{1}{2}} \cos[\nu(q_j - q_k)] \quad (4)$$

$$\frac{d}{dt} p_j = 2\mu\nu p_j^{\frac{1}{2}} \sum_{k=0}^{N-1} p_k^{\frac{1}{2}} \sin[\nu(q_j - q_k)] \quad (5)$$

Then by way of a lengthy analysis, he succeeded to show that this Hamiltonian system is solvable, in the sense that the above evolution equations can be solved in closed form.

The final form of his solution take the following form:

$$q_j(t) = q_j(0) + \lambda N t + \nu^{-1} \arctan \left(\frac{\sin(\nu[\mu N t + \alpha - q_j(0)]) - \sin(\nu[\alpha - q_j(0)])}{\cos(\nu[\mu N t + \alpha - q_j(0)]) - \cos(\nu[\alpha - q_j(0)])} + \frac{N}{A} [p_j(0)]^{\frac{1}{2}} \right) \quad (6)$$

$$p_j(t) = p_j(0) + 2\frac{A}{N}[p_j(0)]^{\frac{1}{2}} \left(\cos(\nu[\mu Nt + \alpha - q_j(0)]) - \cos(\nu[\alpha - q_j(0)]) \right) + 2\left(\frac{A}{N}\right)^2[1 - \cos(\nu\mu Nt)] \quad (7)$$

Where the constants A and α are determined by the initial data as follows:

$$C(0) \equiv A \cos \alpha = \sum_{k=0}^{N-1} p_k(0)^{\frac{1}{2}} \cos[\nu q_k(0)] \quad (8)$$

$$S(0) \equiv A \sin \alpha = \sum_{k=0}^{N-1} p_k(0)^{\frac{1}{2}} \sin[\nu(q_k(0))] \quad (9)$$

This part of the analysis was done mainly by solving the equation of motion for the functions $C(t)$ and $S(t)$ plus lengthy and at times cumbersome use of trigonometry.

He then went on to study the " Behavior Near Equilibria " (only in the case $\lambda = 0$) and by analysing this system along the lines of the standard theory of small oscillations , concluded that [1]:

" In the case $\lambda = 0$ considered here, the general motion is characterized by a completely periodical behavior..." and ".... In view of the remarkable simplicity of the motion characterizing the model, it is natural to conjecture that there also exist a quantal version of this model and perhaps of some of its generalizations which is solvable and features a spectrum whose discrete part is equispaced....". The quantum version of this model was then discussed in ref.[2] .

This part of the analysis was done by heavy use of properties of some special matrices introduced for this purpose in [1] .

The discovery of new integrable systems have always created excitement and great activity in mathematical physics. These models have always proved to have a hidden algebraic structure which is responsible for their integrability [4]. The study of this algebraic structure also helps us to generalize the original model in many different ways. Motivated by the desire to understand the model (2) in this framework we have studied this system from a purely algebraic point of view.

Our result is that when written in terms of suitable coordinates the Hamiltonian (2) takes a very simple form, which is nothing but the Hamiltonian of N Non-Interacting harmonic oscillators. All the results of Calogero [1] and Calogero and van Diejen [2] can then be derived without doing any calculations. Furthermore we will find the integrals of motion which are in involution, hence we prove the integrability of the system and find the explicit form of the action-angle variables .

2 The New Dynamical Variables

Define the variables

$$a_j = \left(\frac{p_j}{\nu}\right)^{\frac{1}{2}} e^{i\nu q_j} \quad (10)$$

$$a_j^* = \left(\frac{p_j}{\nu}\right)^{\frac{1}{2}} e^{-i\nu q_j} \quad (11)$$

Since the parameter ν has the dimension of inverse lenght, the above variables will have the dimension of $(\text{Action})^{\frac{1}{2}}$ and will have the following poisson brackets:

$$\{a_j, a_k\} = \{a_j^*, a_k^*\} = 0 \quad \{a_j, a_k^*\} = i\delta_{j,k} \quad (12)$$

The Hamiltonian will then take the following form:

$$H = N\lambda\nu \sum_{j=0}^{N-1} a_j^* a_j + \mu\nu \sum_{j,k=0}^{N-1} a_j^* a_k \quad (13)$$

which can be put into the compact matrix form:

$$H = N\lambda\nu A^\dagger A + \mu\nu A^\dagger C A \quad (14)$$

where

$$A = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_{N-1} \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 & . & . & . & 1 \\ 1 & 1 & . & . & . & 1 \\ 1 & 1 & . & . & . & 1 \\ . & . & . & . & . & . \\ 1 & 1 & . & . & . & 1 \end{pmatrix} \quad (15)$$

where dot stands for 1. Having all of its elements equal to 1, the matrix C has the following orthonormal eigenvectors:

$$C \mathbf{u}_\alpha = \xi_\alpha \mathbf{u}_\alpha \quad \alpha = 0, 1, 2, \dots, N-1 \quad (16)$$

where

$$\xi_0 = N \quad \xi_1 = \xi_2 = \xi_3 = \dots \xi_{N-1} = 0 \quad (17)$$

and

$$(\mathbf{u}_\alpha)_s = \frac{1}{\sqrt{N}} \omega^{\alpha s} \quad \text{with} \quad \omega = e^{\frac{2\pi i}{N}} \quad (18)$$

i.e:

$$\mathbf{u}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \quad \mathbf{u}_1 = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \cdot \\ \cdot \\ \omega^N \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \cdot \\ \cdot \\ \omega^{2N} \end{pmatrix} \quad \text{etc.} \quad (19)$$

Clearly the matrix R which diagonalizes the hermitian matrix C is unitary, with

$$RCR^\dagger \equiv D = \text{diagonal}(N, 0, 0, 0 \dots 0) \quad (20)$$

where the explicit form of R is given by

$$R_{\alpha,\beta} = (\mathbf{u}_\alpha)_\beta = \frac{1}{\sqrt{N}} \omega^{\alpha\beta}$$

Defining the new variables $B = RA$, i.e:

$$b_\alpha = \frac{1}{\sqrt{N}} \omega^{\alpha s} a_s \quad b_\alpha^* = \frac{1}{\sqrt{N}} \omega^{-\alpha s} a_s^* \quad (21)$$

we find:

$$H = N\lambda\nu B^\dagger B + \mu\nu B^\dagger DB \quad (22)$$

or

$$H = N\lambda\nu\left(b_0^*b_0 + b_1^*b_1 + b_2^*b_2 + \dots + b_{N-1}^*b_{N-1}\right) + \mu N\nu b_0^*b_0 \quad (23)$$

It is essential that the new variables satisfy canonical commutation relations which is a simple consequence of unitarity of R.

$$\{b_\alpha, b_\beta\} = \{b_\alpha^*, b_\beta^*\} = 0 \quad \{b_\alpha, b_\beta^*\} = i\delta_{\alpha,\beta} \quad (24)$$

Defining now the quantities $I_\alpha \equiv b_\alpha^*b_\alpha$ which are global functions of the old coordiantes we find :

$$\{I_\alpha, I_\beta\} = 0 \quad \forall \alpha, \beta \quad (25)$$

The quantities I_α are the N integrals of the motion, in involution with each other and the Hamiltonian is a function of these integrals,i.e:

$$H = N\lambda\nu\left(I_0 + I_1 + I_2 + \dots I_{N-1}\right) + \mu\nu N I_0 \quad (26)$$

Therefore the system (2) is integrable in the Lioville sense. The explicit form of the integrals I_α is found from (10-11) and (21) to be :

$$I_\alpha = \frac{1}{2N\nu} \sum_{k,k'=0}^{N-1} (p_k p_{k'})^{\frac{1}{2}} \cos[\nu(q_{k'} - q_k) + \frac{2\pi\alpha}{N}(k' - k)] \quad (27)$$

These functions , having the dimension of action are in fact the action variables. We can also find the angle variables Q_α . These are:

$$Q_\alpha = \frac{1}{2i} \ln \frac{b_\alpha}{b_\alpha^*} \quad (28)$$

It is straightforward to check from (24) that the action-angle variables have canonical poisson brackets:

$$\{I_\alpha, I_\beta\} = \{Q_\alpha, Q_\beta\} = 0 \quad \{Q_\alpha, I_\beta\} = \delta_{\alpha,\beta} \quad (29)$$

The total momentum P has a simple expression in terms of the action variables:

$$P = \nu \left(I_0 + I_1 + I_2 + \dots I_{N-1} \right) \quad (30)$$

It is now clear that the initial value problem can be solved in very simple and closed form. The equations of motion are:

$$\frac{d}{dt} b_\alpha = i\nu N \lambda b_\alpha \quad \alpha \neq 0 \quad (31)$$

$$\frac{d}{dt} b_0 = i\nu N (\lambda + \mu) b_0 \quad (32)$$

with solutions

$$b_\alpha(t) = b_\alpha(0) e^{iN\lambda t} \quad \alpha \neq 0 \quad (33)$$

and

$$b_0(t) = b_0(0) e^{iN(\lambda+\mu)t} \quad (34)$$

Using the inverse transformation of (21) one obtains the time evolution of the variables (a_i, a_i^*) and hence of the original coordinates and momenta. It is only for the difference between evolution of $b_{\alpha \neq 0}$ and b_0 that the variables (a_i, a_i^*) and hence q_i and p_i have a complicated-looking evolution. Otherwise (in the case $\mu = 0$) one will find that the variables (a_i, a_i^*) have exactly the same simple time evolution as the variable (b_i, b_i^*) .

3 Generalizations

One can now generalize the system (2) in many different ways. In fact from our analysis it is seen that if the matrix C in (14) is replaced by any other hermitian matrix , none of the main results of this paper will change, except the explicit form of the diagonalizing matrix. A good choice of the matrix C which is perhaps much better than (15) on physical grounds , is the following:

$$C = \begin{pmatrix} 0 & 1 & . & . & \sigma \\ 1 & 0 & 1 & . & . \\ . & 1 & 0 & 1 & . \\ . & . & 1 & 0 & 1 \\ \sigma & . & . & 1 & 0 \end{pmatrix} \quad (35)$$

where dot stands for 0. This matrix leads to the following Hamiltonian

$$H = N\lambda\nu \sum_{j=0}^{N-1} a_j^* a_j + \mu\nu \sum_{j=0}^{N-1} a_j^* a_{j+1} + a_{j+1}^* a_j + \mu\nu\sigma(a_1^* a_N + a_N^* a_1) \quad (36)$$

or

$$H = N\lambda \sum_j^N p_j + \mu \sum_{j=0}^{N-1} (p_j p_{j+1})^{\frac{1}{2}} \cos[\nu(q_j - q_{j+1})] + \mu\sigma(p_1 p_N)^{\frac{1}{2}} \cos[\nu(q_1 - q_N)] \quad (37)$$

which represents a system with local (nearest neighbor) interaction , rather the long range interactions implied by (2). Physically, systems with local interactions are much

more interesting than those with long-range interactions. Also it is meaningful to think of (35) as a one dimensional system whereas for the system (2) where all the particles interact with each other the concept of dimensionality is somehow vague. The parameter σ in (34-36) is used to treat systems with various boundary conditions. When $\sigma = 0$ we are dealing with open boundary conditions and when it is non-zero we are dealing with periodic boundary conditions.

4 Discussion

We have shown that the integrable system proposed by Calogero when written in suitable coordinates is in fact a system of non-interacting harmonic oscillators. Our results also show how (in the quantum case) the spectrum of this model can be calculated in a quite simple way. From (24) it is seen that the energy eigenvalues are given by :

$$E = N\lambda\nu(n_0 + n_1 + n_2 + \dots n_{N-1}) + N\mu\nu n_0 \quad (38)$$

where n_i 's are positive integers. At first sight, it may seem disappointing that the solvable systems of [1,2] are in fact non-interacting harmonic oscillators in a different guise. However one should note that after all any *integrable* system when written in terms of the action angle variables will be a *non-interacting* system. Therefore from a rigorous point of view the systems studied in [1,2] deserve to be named " new solvable systems " although they are too easy to solve when treated properly. Other integrable

systems of the form

$$H = \sum_{j,k}^N (p_j p_k) \{ \lambda + \mu \cos[\nu(q_j - q_k)] \}$$

have been proposed by Calogero in [5]. The integrable structure of these systems have been analysed algebraically in [6,7] , where it has been shown that these systems are nothing but a system of spins with long range interaction.

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